Invited Paper

Generation of Random Fractal Fields by a Double Scattering Process

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ABSTRACT

Intensity distributions and their correlation properties are investigated theoretically and experimentally for speckle fields produced by a double scattering process by means of a random fractal object and an ordinary phase screen. Speckle patterns generated in the Fraunhofer diffraction plane (a focal plane of a Fourier-transforming lens) of the second diffuser have clustered and self-similar appearances, and hence are regarded to be fractal. Their intensity correlation functions are shown to obey a power law. The similar intensity distributions and correlation properties are also observed in lateral planes at different distances from the lens. The correlation function in the longitudinal direction is also examined and found to have a power-law behavior, indicating the existence of three-dimensional fractality of the field. Finally, fractality of speckle patterns produced in the Fresnel diffraction region of the ordinary diffuser is demonstrated.

Keywords: Fractal speckle, doubly scattered speckle, power law, intensity correlation, diffraction

1. INTRODUCTION

It is recognized nowadays that various fractal geometries are found in many structures that are formed naturally and artificially. Typical examples are found in physical processes such as phase transitions, aggregations, and interface formations, as well as in biological structures such as neurons and vessel networks.[1,2] To apply the optical technology to such structures with fractal properties, it is desired to understand how the fractality of a given object influences the properties of the diffracted or scattered field. Since the first explicit attempt made by Berry,[3] who coined the word ***diffractals*** to denote such optical fields, extensive studies have been made from this viewpoint in the past two decades.[4,5]

On the other hand, if one consider any applications of the concept of fractal to the optical technology, it is attractive to create an optical field having a fractal property which is controllable in a certain manner. The purpose of this paper is to discuss a method to produce random optical fields that have controllable fractal properties. Such fields will be observed as a kind of speckle patterns having power-law tails in spatial correlation of intensities. As a practical situation for producing such speckles, we take advantage of the fact that the average spatial distribution of the intensity scattered by a fractal object becomes a power function q^{-D}, where D is the fractal dimension of the fractal object and q is the radial coordinate of the observation plane.[1,6] On the other hand, if the intensity distribution of the illuminating spot on an ordinary scattering surface is a power function, the autocorrelation function of speckles observed in the Fraunhofer diffraction region is also expected to take a power-law behavior owing to the relation analogous to the van Cittart-Zernike theorem. These two scattering processes indicate that, if an ordinary diffuser is illuminated by diffractals having a power-law average intensity distribution, the resultant doubly scattered fields may have a power-law correlation property.

In this paper, our previous studies[7,8] on this subject are summarized and, in addition, a recent development is introduced. In Section 2, spatial intensity correlations are discussed theoretically for the field that is scattered first by a random fractal object and subsequently by an ordinary diffuser, and is produced in the Fraunhofer diffraction plane of the ordinary diffuser. It is shown that the resultant intensity correlation obeys a power function under certain conditions. This theoretical prediction is verified experimentally in Section 3, and several peculiar features of such intensity distributions are elucidated. Further experimental consideration is given in Section 4 to explore the three-dimensional fractality around the Fraunhofer diffraction plane of the ordinary diffuser. In Section 5, fractal optical fields produced in the Fresnel diffraction region of the ordinary diffuser are discussed.

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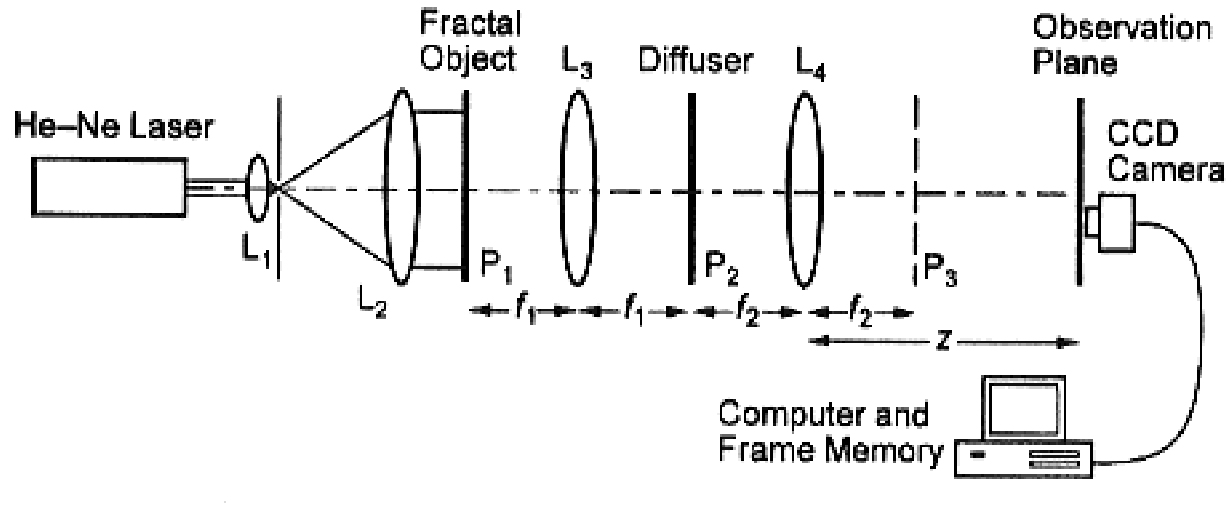


Figure 1. Experimental set-up for producing speckles with fractal property.

2. THEORY

Let us consider an optical system shown in Fig. 1. In the object plane P_1 is placed a random fractal object. Assume that the amplitude transmittance of the fractal object has a property of mass fractals. That is, by identifying a transmittance distribution $T(\boldsymbol{u})$ with a mass distribution, the total transmittance $M_T(L)$ included in a circle of radius L obeys a power function

$$M_T(L) = \int_S T(\boldsymbol{u})d\boldsymbol{u} \propto L^D, \tag{1}$$

where S is the area inside the circle and D is a fractal dimension of the object. As a property of mass fractals, the autocorrelation function of the transmittance distribution is also given by a power law

$$C_T(\boldsymbol{u}) = \langle T(\boldsymbol{u}+\boldsymbol{u}')T(\boldsymbol{u}')\rangle \propto u^{D-d}, \tag{2}$$

where d is the dimension of the Euclidean space where the fractal object is embedded and is $d = 2$ in the present situation.

When this object is illuminated with a coherent plane wave from He-Ne laser as shown in Fig. 1, a speckle pattern is produced in the Fraunhofer diffraction plane P_2, the field of which is given by the Fourier transform of the field just after the fractal object. It follows therefore that the spatial distribution of ensemble-average intensity of the speckle pattern corresponds to the power spectrum of the field in P_1, and is given by

$$\langle I(\boldsymbol{q})\rangle \propto q^{-D}, \tag{3}$$

where $\boldsymbol{q}$ is a coordinate vector in the plane P_2. For a more realistic model, a singularity of $\langle I(\boldsymbol{q})\rangle$ at the origin is avoided by introducing an approximation

$$\langle I(\boldsymbol{q})\rangle = \langle I(0)\rangle\,[1+\alpha^2q^2]^{-D/2}, \tag{4}$$

where $\alpha = (2\pi/\lambda f_1)R$ is a quantity proportional to the maximum scale R of the fractal object.

Next, consider that an ordinary plane diffuser is placed in the plane P_2, and is illuminated by the speckle field which has the average intensity distribution of eq. (4). Let us assume that (i) the field illuminating the second diffuser is a spatially stationary Gaussian speckle pattern with the average speckle size greater than the lateral correlation size of the second diffuser and that (ii) if the second diffuser is uniformly illuminated, the speckle field scattered from the second diffuser obeys the Gaussian statistics in the observation plane. Under these conditions, together with some other easily acceptable conditions, the normalized correlation function of the intensity observed in the Fraunhofer diffraction plane P_3 of the second (ordinary) diffuser is shown to be given, asymptotically for $r \ll MR$, by[7]

$$\begin{aligned} C_I(\boldsymbol{r}) &= \frac{\langle I(\boldsymbol{r}_1)I(\boldsymbol{r}_2)\rangle - \langle I(\boldsymbol{r}_1)\rangle\langle I(\boldsymbol{r}_2)\rangle}{\langle I(\boldsymbol{r}_1)\rangle\langle I(\boldsymbol{r}_2)\rangle} \\ &\propto \begin{cases} (r/MR)^{2(D-2)} & \text{for} \quad 1<D<2, \\ [\log(r/MR)]^2 & \text{for} \quad D=2, \\ 1 & \text{for} \quad 2<D<3, \end{cases} \end{aligned} \tag{5}$$

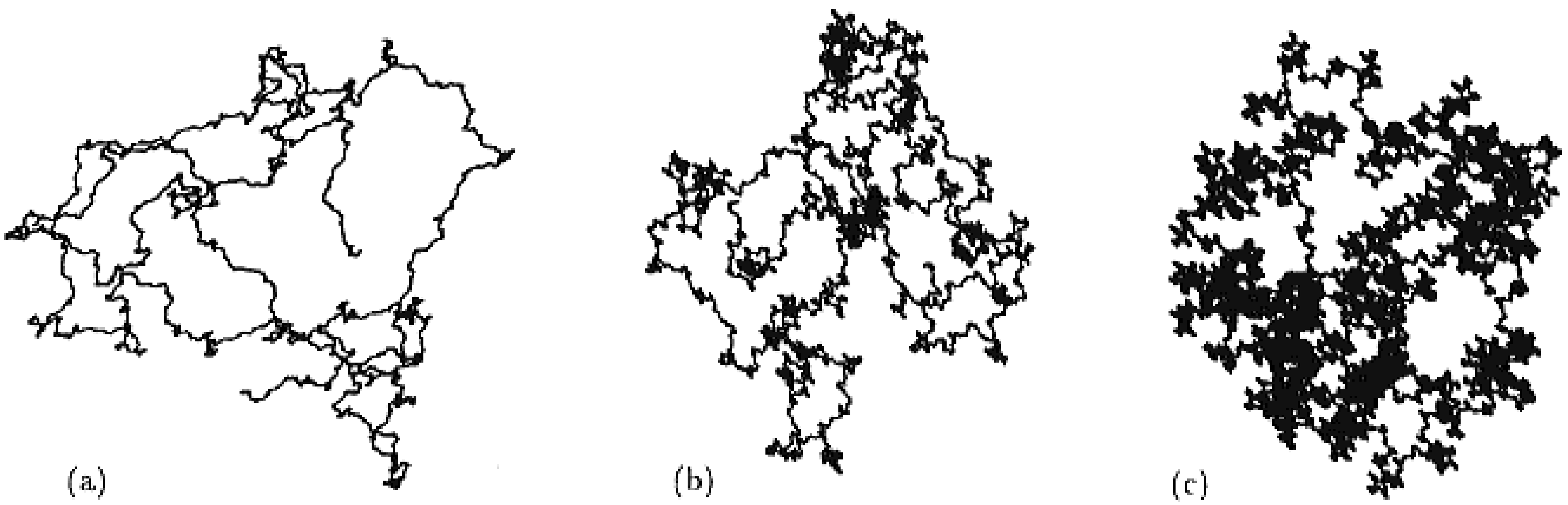


Figure 2. Random fractal objects produced by using the band-limited Weierstrass functions and employed in experiments. Assigned fractal dimension was $D =$ (a) 1.2, (b) 1.5, and (c) 1.8.

where $M = f_2/f_1$ is a magnification of the optical system. Equation (5) shows that the intensity correlation function obeys a power law for $1 < D < 2$, and this indicates that the speckle pattern observed in the plane P_3 is fractal. It should be noted here that the parameter α used in eq. (12) and later in Ref. [7] should be interpreted as $\alpha = MR$.

3. EXPERIMENT: FRAUNHOFER DIFFRACTION PLANE

3.1. Experimental configuration

In the present study, we employ random fractal objects having different fractal dimensions D generated by band-limited Weierstrass functions.[9] For the present purpose, two Weierstrass functions defined by

$$\begin{aligned} x(t) &= \eta \sum_{n=0}^{N} a^n \cos(sb^n t + \phi_n) \\ y(t) &= \eta \sum_{n=0}^{N} a^n \cos(sb^n t + \phi'_n) \end{aligned} \tag{6}$$

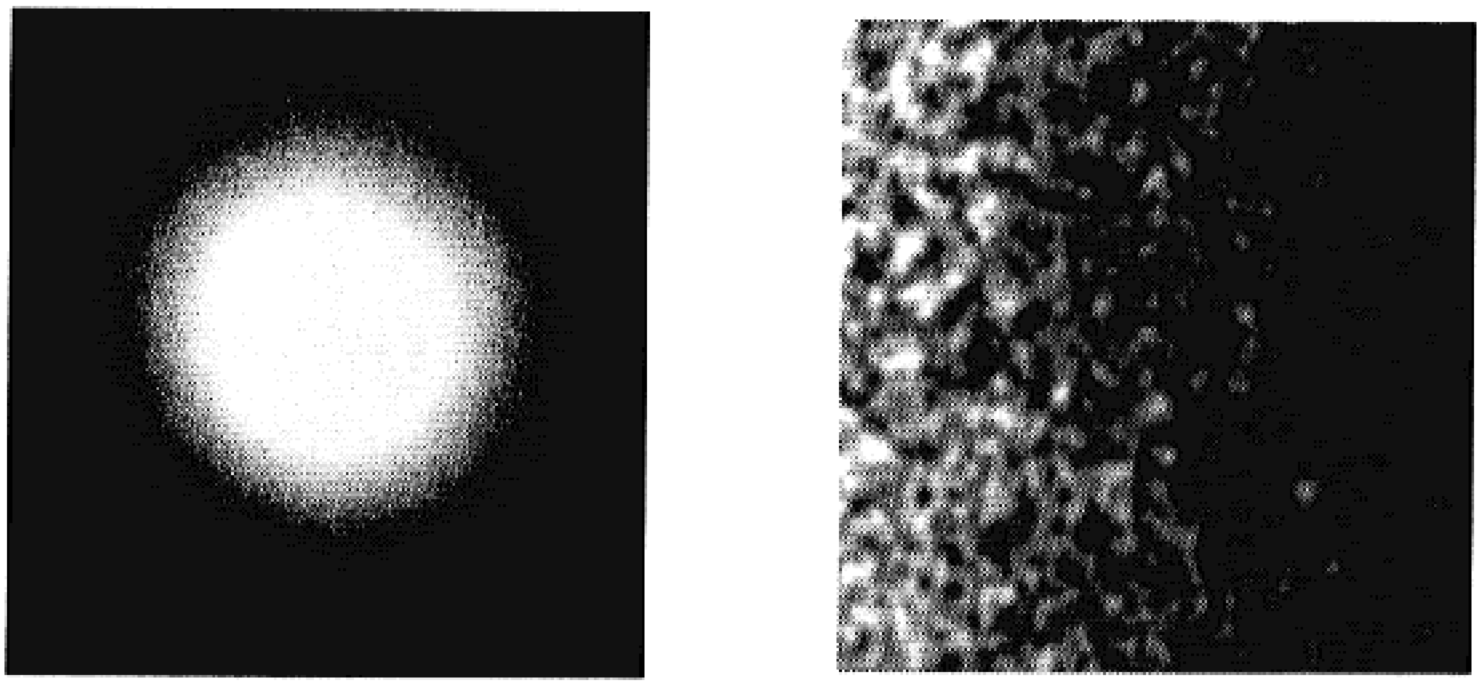
are combined to generate a fractal trail $(x(t), y(t))$. In eq. (6), η is a perturbation amplitude, s is a scaling factor, ϕ_n and ϕ'_n are uniform random variables over the interval $[0, 2\pi]$, and N is the number of tones with the relative spacing controlled by b. The fractal dimension of the trail is given by $D = \log b / \log a$.[9,10] Actually, the trail was calculated for discrete values of t. Examples of the generated trails are shown in Fig. 2. In generating the trails in this figure, the number of tones was set to be 20 while the associated fractal dimension was $D =$ (a) 1.2, (b) 1.5, and (c) 1.8. These random fractal objects were printed out by a laser printer and were subsequently photographed on films, which were used as the fractal objects.

With reference to Fig. 1, the speckle field produced by the fractal object was incident on a ground glass plate placed at the back focal plane P_2 of L_3. The speckle pattern formed by the ground glass plate was recorded by a CCD camera at the back focal plane P_3 of a lens L_4 and was stored in a computer through a frame memory as discrete data with 256 levels.

For comparison with the speckled speckles produced by the configuration of Fig. 1, an ordinary type of speckle pattern was also recorded by the following configuration: The object at P_1 and the lens L_3 were removed and the circular aperture was set at P_2 to mask directly the diffuser. Then, the diffuser was illuminated normally with a plane wave and a typical speckle pattern was produced at P_3.

3.2. Intensity distributions

When the fractal object is placed at P_1, the speckle pattern illuminating the diffuser has an average intensity obeying the power-law given by eq. (4). An experimental result of this pattern for the object of $D = 1.8$ is shown in Fig.



(a) (b)

Figure 3. (a) Fraunhofer diffraction pattern of the random fractal object with $D = 1.8$ and (b) its magnified portion exhibiting the speckles.

3(a). In taking this pattern, a central portion was intentionally overexposed to make its decreasing region visible. Speckles of this figure are much more visible in Fig. 3(b) which is a magnified portion of the pattern in Fig. 3(a).

Figures 4(a)–(c) show the speckled speckles observed at P_3 using the fractal objects of $D =$ (a) 1.2, (b) 1.5, and (c) 1.8. A remarkable feature common to these figures is that they look quite different from ordinary speckle patterns (see Fig. 5) in the sense that they have no definite speckle size. Instead, these patterns have lumps or clusters of intensity with various scales. This appearance strongly implies the presence of fractality in these patterns. It is also noted that the size of the intensity cluster tends to increase with an increase in the dimension of the fractal object, implying an even longer correlation for a larger fractal dimension of the object. Figures 4(d), (e) and (f) show twice magnified portions of (a), (b) and (c), respectively. Each of the magnified patterns looks similar in a statistical sense to the corresponding original pattern, and this indicates the statistical self-similarity of these intensity distributions.

For comparison, a speckle pattern produced by the modified configuration described above is shown in Fig. 5. This pattern corresponds to the most typical speckle pattern and has a well-defined speckle size. This fact is easily recognized by comparing the pattern of Fig. 5(a) with its twice magnified portion shown in Fig. 5(b).

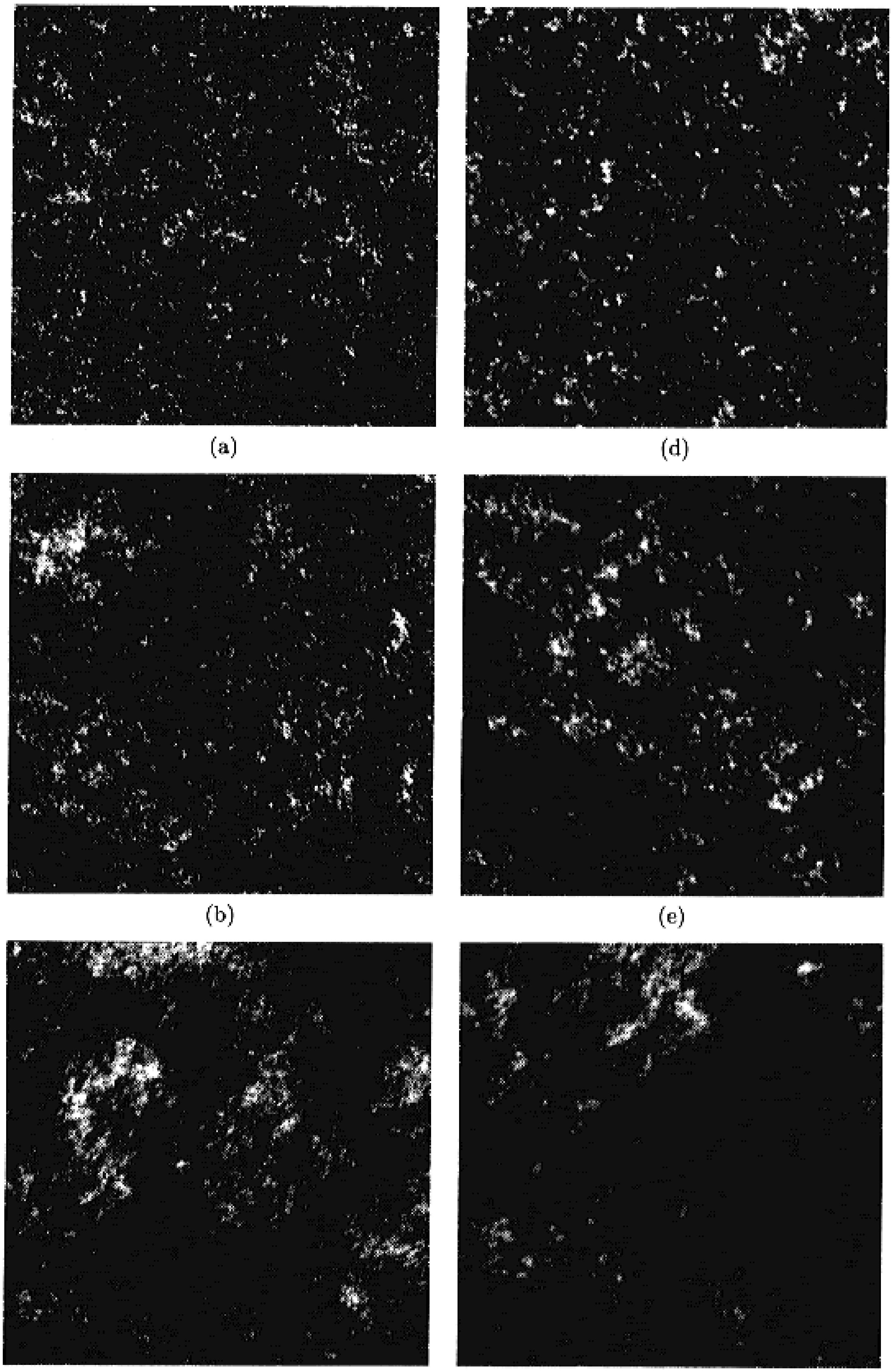
Being different from the conventional speckles shown in Fig. 5, the speckle patterns in Fig. 4 remind us of non-Gaussian speckles, which are also known to exhibit a clustering of speckles under certain conditions.[11] It is therefore interesting to examine first order statistics of the observed speckles. To this end, probability densities were actually calculated from the speckle intensities and were found to obey negative exponential density. This means that the speckles shown in Figs. 4 and 5 are in the Gaussian regime and obey zero-mean circular complex Gaussian statistics.

3.3. Intensity correlations

To examine fractality of the speckles, we computed autocorrelation functions of the speckle intensities. Since the patterns in Figs. 4 and 5 are statistically isotropic due to an isotropy of the employed objects, we derived an angular-averaged correlation $C_I(r)$ from the two-dimensional correlation function $C_I(\mathbf{r})$. The results are shown logarithmically in Fig. 6.

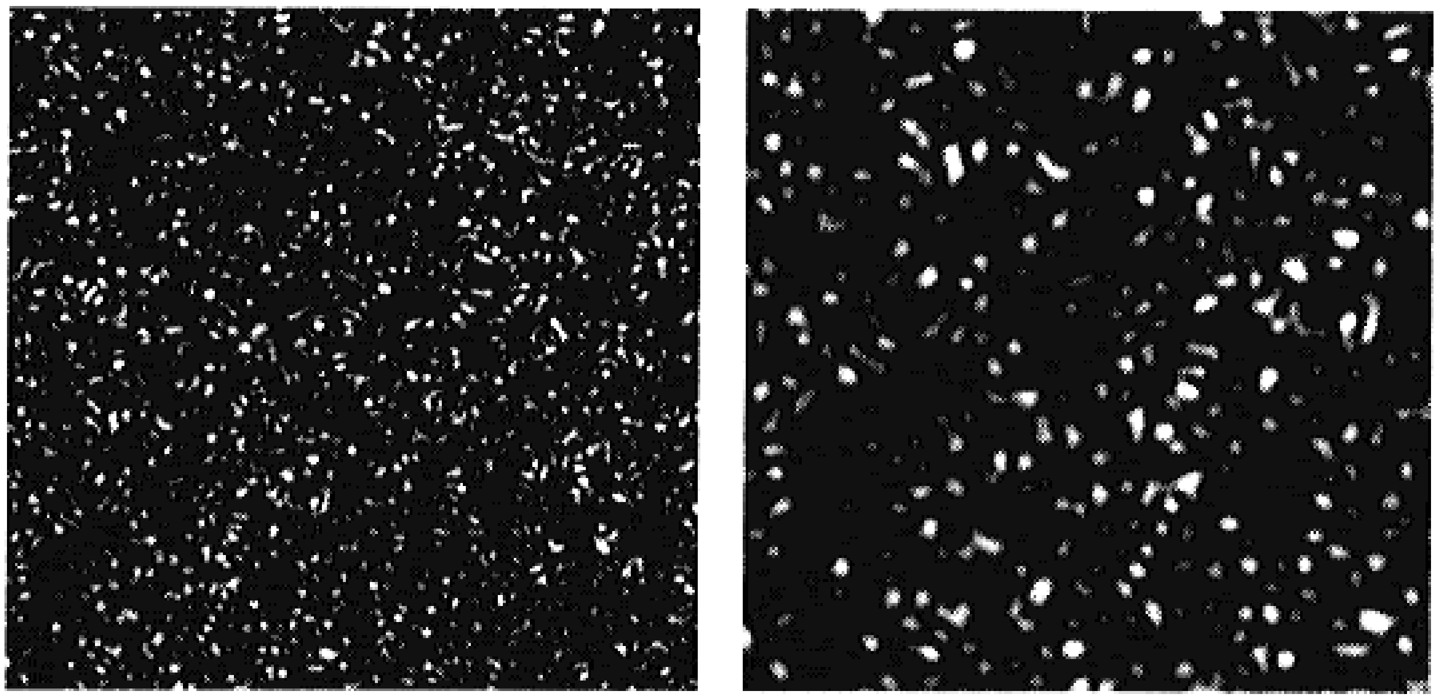
The intensity correlations denoted by A–C in Fig. 6 are for the speckled speckles due to the fractal objects, and are approximately linear for $r < 30$ mm in a logarithmical plot. This means that the correlation functions are nearly power functions as expected from eq. (5) in this range of r. Hence, these speckles can be regarded as fractals at least for the first approximation.

On the other hand, the correlation function labeled D corresponding to the pattern of Figs. 5 has a rather flat region for small r with relatively clear cutoff beyond that region, in accordance with the existence of well-defined speckle size. By contrast with this curve having the cutoff, the singular characteristic of the curves A–C is evident.



(a) (d) (b) (e) (c) (f)

Figure 4. Speckled-speckle patterns produced by using the fractal objects with the fractal dimension of $D = 1.2$ (a) and (d); 1.5 (b) and (e); and 1.8 (c) and (f). (d), (e) and (f) are twice magnified portions of (a), (b) and (c), respectively, showing the statistically self-similar appearance.



(a) (b)

Figure 5. (a) Ordinary speckle pattern produced by removing the first object and the lens L_3 and using a circular aperture to mask directly the diffuser, and (b) a magnified portion of (a) by the factor of 2.

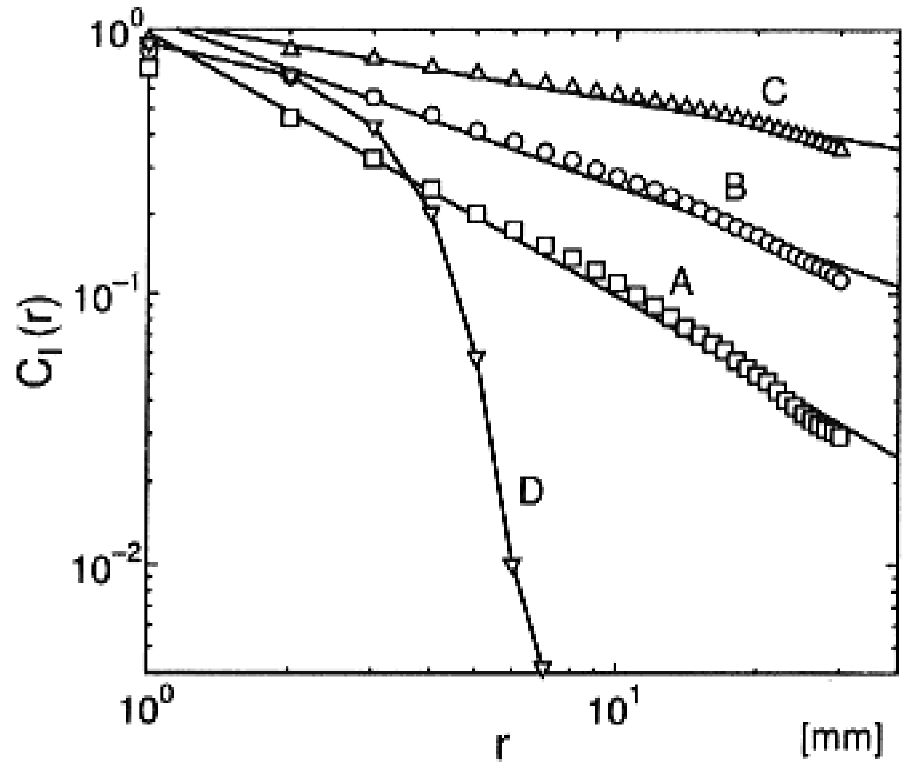


Figure 6. Angular-averaged intensity correlation functions $C_I(r)$ of speckles produced by the random fractal objects of $D =$ (A) 1.2, (B) 1.5, and (C) 1.8, and by (D) the circular aperture set in contact with the diffuser without the first object and L_3.

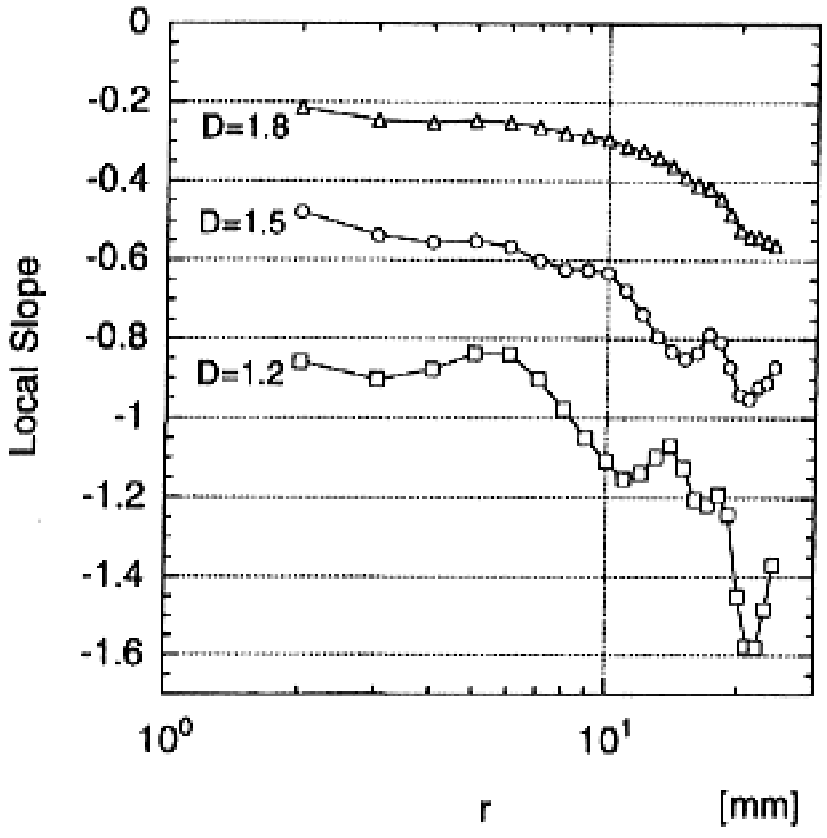


Figure 7. Local slopes of the logarithmically plotted intensity correlation functions $C(r)$ for three random fractal objects.

It is noted, however, that the curves A–C are not straight lines but rather tend to bend downward slightly with an increase in r. This is seen as deviations of the experimental points from their least-square fits shown by the solid lines. This tendency makes it difficult to determine the global slopes of the curves and, therefore, we calculated local slopes from every three successive points in each curve. The results for the three curves are shown in Fig. 7, which may be compared with the theoretical slopes predicted by eq. (5); $2(D-2) = -1.6, -1$, and -0.4 for $D = 1.2$ (A), 1.5 (B), and 1.8 (C), respectively. The experimental values shown in Fig. 7 are a little larger than the theoretical predictions in most points, but have decreasing trends, approaching the theoretical values for larger r in the range shown. It is noted that the local increases of the local slopes for $D = 1.2$ and 1.5 near the larger end of r is not a significant trend, but is simply due to growing statistical fluctuations associated with the decreasing correlation values for larger r.

3.4. Fractal dimension of the speckle pattern

The slope of the logarithmically plotted correlation function, i. e. the exponent of the power function, is an important quantity for the characterization of fractal. To discuss about this quantity, we have to clear the problem concerning the correlation function of fractals. As is shown in eq. (2), an isotropic spatial distribution of any quantity $\rho(\boldsymbol{r})$ with mass fractal property has a spatial correlation of

$$C_\rho(r) = \langle \rho(\boldsymbol{r}+\boldsymbol{r}')\rho(\boldsymbol{r}')\rangle \propto r^{D_s-d}, \tag{7}$$

where D_s is the fractal dimension of ρ. However, the correlation function $C_I(r)$ of eq. (5) has a different definition from eq. (7) in such a way that the product of the average intensities is subtracted in the former. This difference makes us hesitate in concluding that the speckles satisfying eq. (5) are fractal. However, this difficulty can be overcome in the following discussion.

The correlation in the form of eq. (7) is obeyed typically by fractal mass distributions with the non-negative mass density. Nevertheless, eq. (7) requires that

$$C_\rho(r) \to 0, \quad \text{as} \quad r \to \infty. \tag{8}$$

This is possible only for ideal fractal distributions having an unlimited spatial extent. For such fractals, unlimitedly large vacancies can exist for $r \to \infty$, which gives rise to $\rho(r) \to 0$, leading to a vanishing correlation. In other words, the average $\langle\rho(\boldsymbol{r})\rangle$ vanishes in the limit of an infinitely large fractal distribution, since the fractal dimension D_s is always smaller than the Euclidean dimension d of the space. In any practical fractal distributions, however, the correlation never vanishes due to a finite extent of the fractality, but rather reaches the finite value given by $\langle\rho(\boldsymbol{r})\rangle^2$ for $\boldsymbol{r}$ exceeding the extent of the fractality. This corresponds to the term $\langle I(\boldsymbol{r})\rangle^2$ to be subtracted from the speckle correlation.

It is also noted that, for actual distributions, the peak value of the correlation is also finite contrary to the theoretical correlation of eq. (7). Therefore, a practical form for C_ρ crossovers from the power law to a certain finite value as r approaches 0. In the case of Gaussian speckles, there is an additional restriction for the correlation function. For such speckle intensities that obey the negative exponential density, we have

$$C_I(0) = \frac{\langle I^2(\boldsymbol{r})\rangle - \langle I(\boldsymbol{r})\rangle^2}{\langle I(\boldsymbol{r})\rangle^2} = 1, \tag{9}$$

which is actually the speckle contrast.[12] This property of Gaussian speckles restrict the correlation function to take values only in the limited range.

With the above considerations in mind, we can invoke eq. (7) to relate the exponent in the intensity correlation function to the fractal dimension of the speckle pattern. By comparing two exponents in eqs. (5) and (7) and by noting $d = 2$ in the present situation, we find a theoretical expression for the fractal dimension D_s of the speckle pattern to be $D_s = 2D - 2$, which gives $D_s =$ (a) 0.4, (b) 1, and (c) 1.6 for the objects of $D =$ (a) 1.2, (b) 1.5, and (c) 1.8, respectively. If the experimental slopes of Fig. 7 are taken into account, the experimental values for D_s for Figs. 4(a)–(c) would be a little smaller than these theoretical predictions.

4. THREE-DIMENSIONAL CORRELATION PROPERTIES

4.1. Lateral correlations at different propagation distances

Next, let us extend the observation plane of the preceding experiment to a three-dimensional region around the focal plane to see whether the fractality of the field has a three-dimensional extent or not. That is, with reference to Fig. 1, we consider an observation plane not restricted to the focal plane P_3 of L_4, but located at some distance z from L_4.

We first examined speckles produced by the three fractal objects at some distances other than the focal distance f_2. The obtained speckles are shown in Fig. 8 for nine combinations of the parameters z and D; $D =$ (a) 1.2, (b) 1.5, and (c) 1.8 for $z = 10$ cm (focal plane): $D =$ (d) 1.2, (e) 1.5, and (f) 1.8 for $z = 50$ cm; and $D =$ (g) 1.2, (h) 1.5, and (i) 1.8 for $z = 150$ cm. Hence, Figs. 2(a)–(c) have the same parameters as Fig. 4.

It is noted from Fig. 8 that the speckle patterns do not change much statistically with the propagation distance z in the explored range. Namely, the speckles seem to depend only on the fractal dimension D of the first scattering object, and not on the propagation distance z. From this observation, it is expected that the intensity correlation function across the lateral observation planes does not change with z.

To verify this quantitatively, the intensity correlation functions $C_I(r)$ were calculated from the observed speckle patterns of the nine combinations of the parameters shown in Figs. 8(a)–(i), and were found to take substantially the same power law for different distances for each value of D, though the correlation values seem to decrease slightly with increasing z.

4.2. Longitudinal correlations

Next, we examine if these speckle fields have a fractality in the longitudinal direction as well as in the lateral directions. To this end, the speckles were recorded at different longitudinal distances from $z = 11$ to 21 cm with a step interval of 2 cm, and also at $z = 10$, 10.1, and 10.5 cm. From these data, the longitudinal correlations were calculated and are shown in Fig. 9. Each data point was calculated from the two speckle patterns: one was fixed at $z_1 = 10$ cm as a reference pattern and the other was chosen from the patterns observed at different longitudinal distances z_2. We see that the three experimental curves in this plot is approximately linear though the approximation is not so good as the case of the lateral correlations shown in Fig. 6. The important point is, however, that the longitudinal correlations of the present speckle fields have much longer tails than expected for ordinary speckles and, hence, we can conclude that the speckled speckles produced by this experiment have a fairly high fractality in the longitudinal direction as well as the lateral direction. It is also interesting to noted that the slopes of the linear

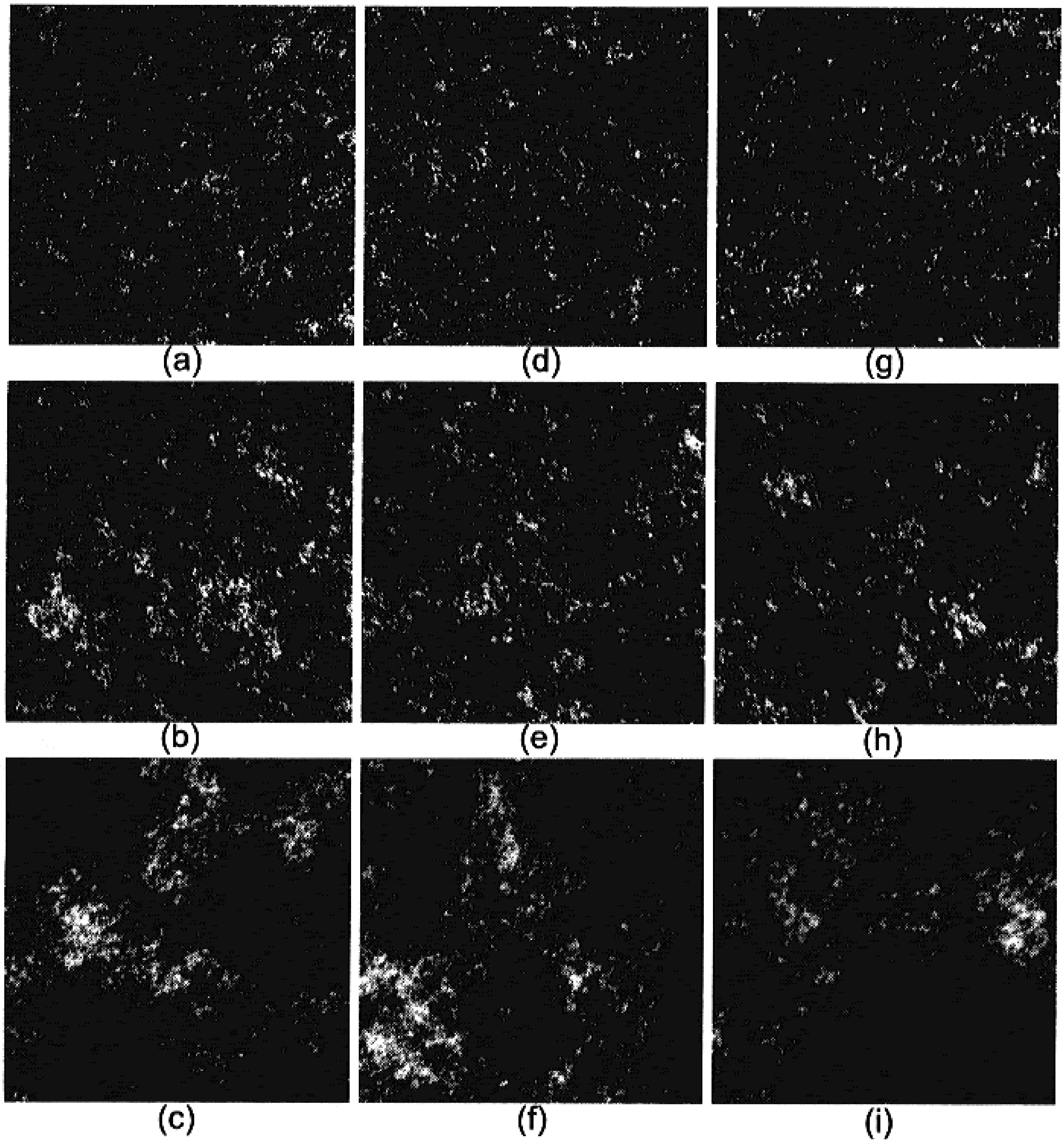


Figure 8. Speckled-speckle patterns observed at longitudinal distances of z = (a)–(c) 10, (d)–(f) 50, and (g)–(i) 150 cm, and for different fractal dimensions of D = (a),(d),(g) 1.2; (b), (e), (h) 1.5; and (c), (f), (i) 1.8.

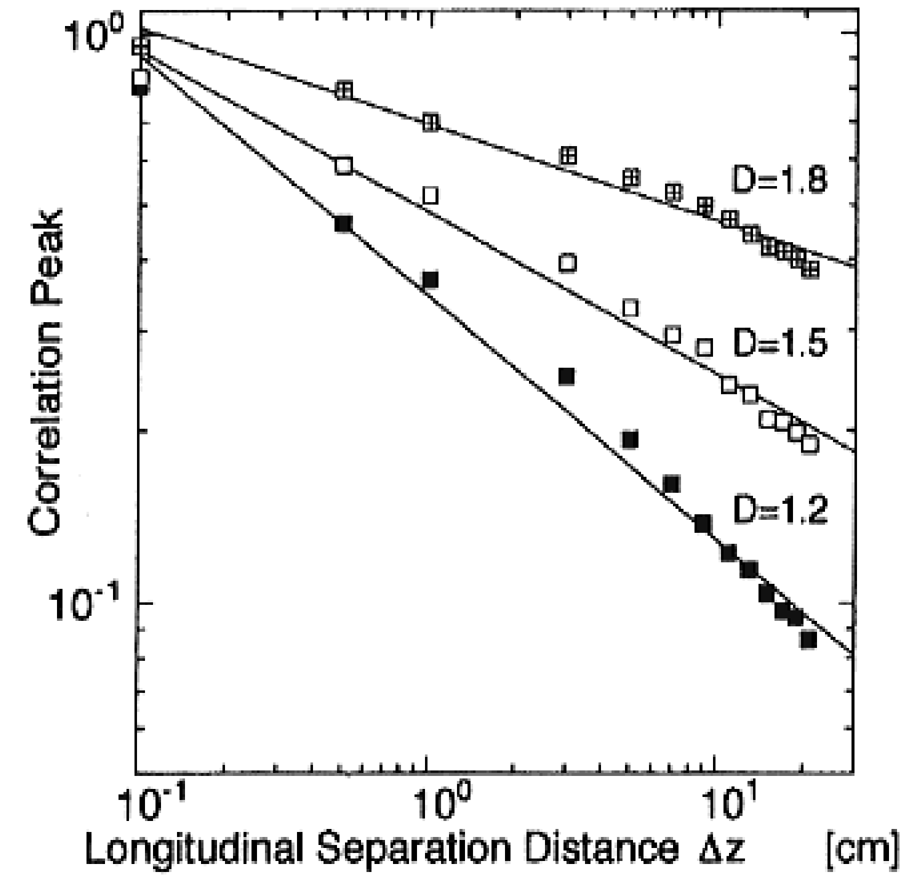


Figure 9. Logarithmical plot of the longitudinal intensity correlations.

dependence in Fig. 9 are more gentle than the corresponding dependence of lateral correlation in Fig. 6. This means that the longitudinal fractal dimension is larger than the lateral fractal dimension for each value of D. Hence, it can be said that the intensity distribution around the focal plane has a three-dimensional fractality with a uniaxial anisotropy.

5. FRESNEL DIFFRACTION REGION

The defocused region as examined in the previous section corresponds to the Fresnel diffraction region of the second diffuser. In order to see the fractal property in the Fresnel diffraction region more directly, we performed a further experiment of measuring intensity distributions by removing the lens L_4 in Fig. 1. That is, scattered light from the second diffuser is directly detected by the CCD camera as a function of the propagation distance z from the diffuser.

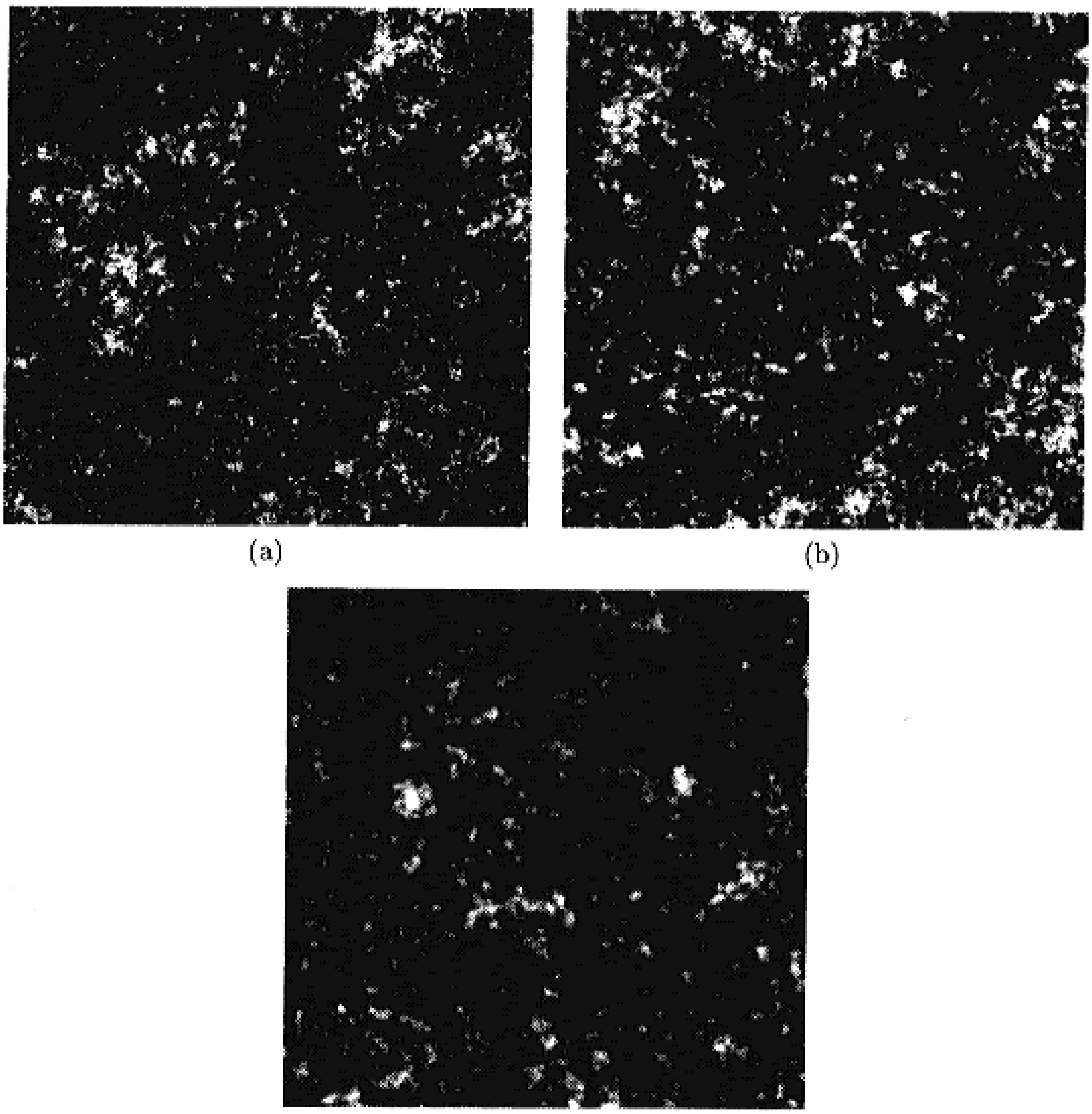
Three speckle patterns obtained by this configuration are shown in Fig. 10 for three distances of (a) 75, (b) 150 and (c) 300 mm from the diffuser. It is seen from this figure that the patterns have similar appearance in a statistical sense apart from fine structures missing gradually with an increase of z. The intensity correlation functions derived from the patterns in Fig. 10 are shown in Fig. 11. In Fig. 11, we can see that the slope of the linearities does not change much with the propagation distance and is also same with that for the Fraunhofer diffraction plane. This result indicates that the fractal property of the doubly scattered intensity distributions is maintained for a fairly wide region including the Fresnel and Fraunhofer diffraction planes.

6. CONCLUSIONS

Theoretical and experimental studies on the generation of random intensity distributions with fractal properties are summarized.

Theoretical background was reviewed briefly to show that, when an ordinary diffuser is illuminated by the Fraunhofer diffraction pattern of a random fractal object, autocorrelation functions of intensity distributions in the Fraunhofer diffraction region of the ordinary diffuser take the form of a power function as an asymptotic behavior for sufficiently small separation distances.

This theoretical prediction was verified experimentally using three fractal objects having the fractal dimension of $D = 1.2$, 1.5, and 1.8 generated with band-limited Weierstrass functions. Some interesting features of the generated fractal intensity distributions, such as intensity clustering, lack of definite speckle size, and statistical self-similar



(a) (b) (c)

Figure 10. Speckled-speckle patterns observed in the Fresnel diffraction region of the second diffuser by using the fractal object of $D = 1.49$. Observation distances are (a) 75, (b) 150 and (c) 300 mm from the diffuser.

appearance, were revealed. The fractal dimension D_s of the speckle patterns obtained was discussed on the basis of the general correlation properties of fractal distributions and was shown to be $D_s = 2D - 2 = 0.4$, 1, and 1.6 for the objects employed.

For the purpose of comparison, ordinary speckle patterns with definite speckle size were also produced and were shown not to obey a power law but to have clear cutoff of correlation. For all the above speckle patterns produced, the probability densities of intensity distributions were found to obey a negative exponential density, indicating that the speckle fields obey zero-mean circular Gaussian statistics.

Intensity correlations were also examined in the three-dimensional space around the Fraunhofer diffraction plane of the second diffuser, and it was found that the fractality extends three-dimensionally with different fractal dimension in the axial direction. Fractal speckle patterns were also observed in the Fresnel diffraction region of the second diffuser and the corresponding intensity correlation functions were shown to obey almost the same power-law in the range of propagation distances examined.

Finally, it is mentioned that the power laws appearing in various properties of fractals are manifestations of the widespread distributions of associated quantities. Namely, the power-law intensity correlation implies that there are widerange of correlations in that intensity distributions from very short correlation components to very large ones.

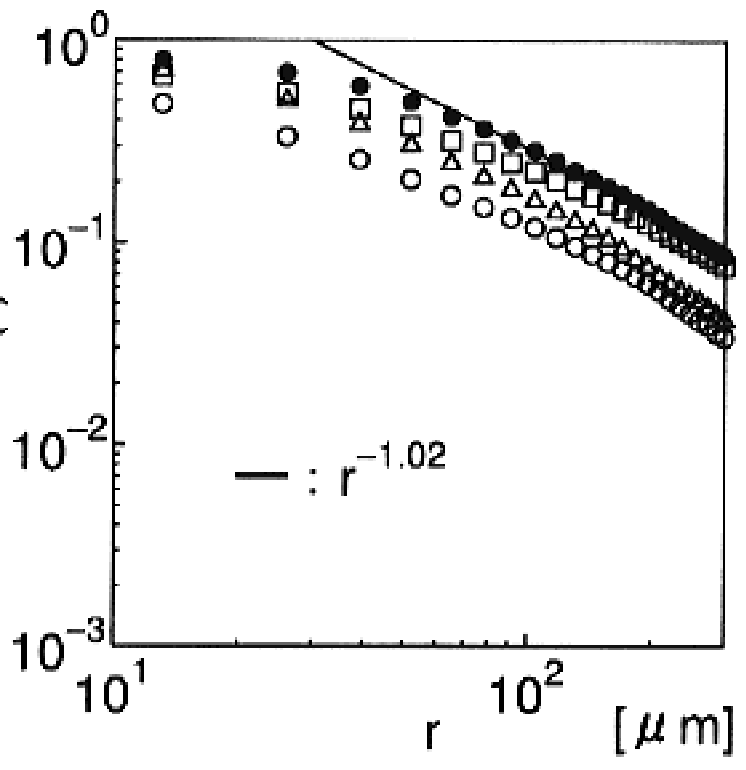


Figure 11. Angular-averaged intensity correlation functions $C_I(r)$ of speckles produced by random fractal object of $D = 1.49$ in the Fresnel diffraction region of the second diffuser. Observation distances are $z = 75$ (○), 150 (△) and 300 mm (□). Data for the Fraunhofer diffraction plane are also plotted with •.

Such properties will provide some advantages to optical techniques based on correlation properties of optical fields. Applications of fractal optical field in such a field are expected.

REFERENCES

1. T. Vicsek, *Fractal Growth Phenomena*, World Scientific, Singapore, 1992.
2. H. Takayasu, *Fractals in Physical Science*, Manchester University, Manchester, 1990.
3. M. V. Berry, "Diffractals," *J. Phys. A: Math. Gen.* **12**, pp. 781–797, 1979.
4. J. Uozumi and T. Asakura, "Fractal Optics" in *Current Trends in Optics*, J. C. Dainty, ed., Academic, London, pp. 83–93, 1994.
5. J. Uozumi and T. Asakura, "Optical Fractals" in *Optical Storage and Retrieval — Memory, Neural Networks, and Fractals*, F. T. S. Yu and S. Jutamulia, ed., Marcel Dekker, New York, pp. 283–320, 1996.
6. J. Feder, *Fractals*, Plenum, New York, 1988.
7. K. Uno, J. Uozumi and T. Asakura, "Correlation properties of speckles produced by diffractal-illuminated diffusers," *Opt. Commun.* **124**, pp. 15–22, 1996.
8. J. Uozumi, M. Ibrahim and T. Asakura, "Fractal speckles," *Opt. Commun.* **156**, pp. 350–358, 1998.
9. D. L. Jaggard and X. Sun, "Scattering from fractally corrugated surfaces," *J. Opt. Soc. Am. A* **7**, pp. 1131–1139, 1990.
10. R.F. Voss, "Random fractals: self-affinity in noise, music, mountains, and clouds," *Physica D* **38**, pp. 362–371, 1989.
11. G. Parry, P.N. Pusey, E. Jakeman and J.G. McWhirter, "Focusing by a random phase screen," Opt. Commun. **22**, pp. 195–201, 1977.
12. J. W. Goodman, "Statistical properties of laser speckle patterns," in *Laser Speckle and Related Phenomena*, 2nd ed., J. C. Dainty, ed., Springer, Berlin, 1984.