Weights Representation of Analytic Hierarchy Process by use of Sensitivity Analysis.

Shin-ichi Ohnishi
Faculty of Engineering,
Hokkai-Gakuen University,
Sapporo, Japan
ohnishi@eli.hokkai-s-u.ac.jp

Hideyuki Imai
Graduate School of Engineering,
Hokkaido University,
Sapporo, Japan
imai@main.eng.hokudai.ac.jp

Takahiro Yamanoi*
Faculty of Medicine
Aix-Marseille II University,
Marseille, France
yamanoi@eli.hokkai-s-u.ac.jp

Abstract

Analytic Hierarchy Process (AHP) is one of major methods for a domain of decision making. However, it often occurs that data of AHP loses its reliability, because a comparison matrix of the data does not always have consistency. In these cases, we suppose that answers from a decision-maker have ambiguous or fuzziness and that weights also have ambiguous or fuzziness. Therefore, it is natural to represent these weights by use of fuzzy sets. The authors propose a new representation of weights (fuzzy weights) by using results from the sensitivity analysis, and present a numerical example. The study shows how results of AHP have fuzziness, when a comparison matrix has relatively not so good consistency.

1 Introduction

Analytic Hierarchy Process (AHP) was proposed by Saaty, T.L. in 1977 [1], [2]. His method has been widely used in a domain of decision making, since it includes the natural feelings of human. However, when we use AHP, it often occurs that a comparison matrix does not have good consistency, since the method contains many activities of decision. In these cases, we consider that answers from decision-makers (i.e. components of comparison matrix) have ambiguity or fuzziness and that weights also have ambiguity or fuzziness. Therefore, it is necessary to represent these weights by use of fuzzy sets.

On the other hand, to analyze how much the components of a pairwise comparison matrix influences weights and consistency of a matrix, we can apply sensitivity analysis to AHP. This allows us possible to know how fuzzy their weights are.

In this paper, the authors propose a representation of weights (fuzzy weights). They are represented as L-R fuzzy numbers by use of the result from the sensitivity analysis. The paper treats not only how to represent weights by fuzzy sets, but also a representation of fuzziness of results from AHP, when a comparison matrix is not always consistent. At last, the paper treats a numerical experiment there by.

2 Summary of AHP

2.1 Process of AHP

(Process 1) Representation of structure by a hierarchy. The problem that we consider is represented by use of a hierarchical structure in Figure 1. The highest level of the hierarchy consists of a unique element that is overall objective. At the

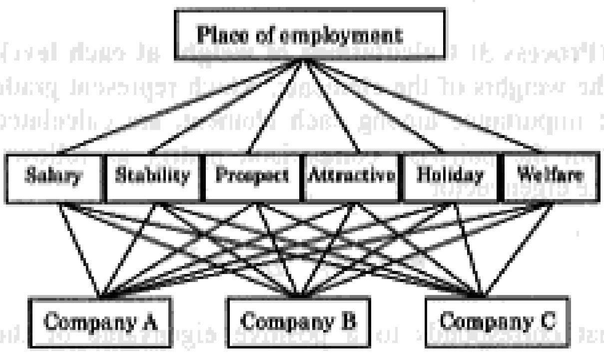

Figure 1: A hierarchical structure

* On leave from Hokkai-Gakuen University

lower level, there are plural activities (i.e. elements at the level) to be considered on their relationships among elements of one level higher, and they are evaluated from subjective judgments of a decision maker. Elements that lie at upper level are called parent elements, and elements that lie at lower level are called child elements. Alternative elements are put at the lowest level of the hierarchy.

(Process 2) Paired comparison between elements at each level. A pairwise comparison matrix A is made from decision maker's answers. Let n be a number of elements at a certain level. The upper triangular components of the comparison matrix a_{ij}, ($i< j = 1,\ldots,n$) are 9, 8, .., 2, 1, 1/2, …, or 1/9. They denote intensities of importance from activity i to j as shown in Table1. About the lower triangular components a_{ji}, we describe reciprocal numbers as follows

$$a_{ji} = 1/a_{ij}, \tag{1}$$

and about diagonal elements, let $a_{ii} = 1$. The lower triangular components and diagonal elements are sometimes omitted to write, because they are evident if upper triangular components are written. The decision maker should make $n(n-1)/2$ paired comparisons at a level with n elements.

Table 1: Components of the comparison matrix

a_{ij}	Definition
1	i is as important as j
3	i is slightly more important than j
5	i is essentially more important than j
7	i is demonstratively more important than j
9	i is absolutely more important than j
2,4,6,8	intermediate values between adjacent scale value
Reciprocals of above	j is more important than i as each above intensity

(Process 3) Calculations of weight at each level. The weights of the elements, which represent grade of importance among each element, are calculated from the pairwise comparison matrix as follows. The eigenvector

$$w = (w_i), \quad \sum_i^n w_i = 1$$

that corresponds to a positive eigenvalue of the matrix is used in calculations throughout in this paper.

(Process 4) Priority of an alternative by a composition of weights. Let w_{li} denote the weight of the element i at the level l, and let u_{ij} be the weight of the child element j at level l+1 with respect to the parent element i. The composite weight of element j of level l+1 can be calculated as follows

$$w_{l+1,j} = \sum_{i \in F_j} w_{li} u_{ij}, \tag{2}$$

where F_j is the set of all parent elements of j. By repeating this, the weights of the alternative, which is the priorities of the alternatives with respect to the overall objective, are finally found.

2.2 Consistency index to the pairwise comparison matrix

Since components of the comparison matrix are obtained by comparisons between two elements, it is not guaranteed its coherent consistency. In AHP, the consistency of the comparison matrix A is usually measured by the following consistency index (C.I.)

$$\text{C.I.} = \frac{\lambda_A - n}{n-1}, \tag{3}$$

where n is the order of matrix A, and λ_A is its maximum eigenvalue.

It should be noted that C.I.$\geqq$0 holds. If the value of C.I. becomes smaller, then the degree of consistency becomes higher, and vice versa. It is said that if the comparison matrix is consistent,

$$\text{C.I.} \le 0.1$$

holds.

3 Sensitivity Analysis of AHP

When we actually use AHP, it often occurs that a comparison matrix is not consistent or that there is not great difference among the overall weights of the alternatives. Thus, it is very important to investigate how components of a pairwise comparison matrix influence on its consistency or on weights. So as to analyze how results are influenced when a certain variable has changed, we use the sensitivity analysis. On the basis of the reasons mentioned above, it is necessary to establish a sensitivity analysis of AHP.

In this paper, we use the method that some of the present authors have proposed before [7]. It evaluates a fluctuation of the consistency index and weights, when a comparison matrix is perturbed, and it is useful because it does not change a structure of the data.

The evaluations of consistency index and the weights of a perturbed comparison matrix are performed as follows.

(1) Perturbations $\varepsilon\ a_{ij}d_{ij}$ are imparted to component a_{ij} of a comparison matrix, and the fluctuation of the consistency index and the weight are expressed by the power series of ε.

(2) Fluctuations of the consistency index and the weights are represented by the linear combination of d_{ij}.

(3) From the coefficient of d_{ij}, it found that how the component of the comparison matrix gives influence on the consistency index and the weight.

Since a pairwise comparison matrix A is a positive square matrix, the following Perron- Frobenius theorem[4] holds.

Theorem 1 (Perron – Frobenius) *About a positive square matrix A, the followings hold.*

1. *Matrix A has a positive eigenvalue. If λ_A is the greatest among them, λ_A is a simple root. The positive eigenvector w_1 corresponding to λ_A, exists. Here λ_A is called the Frobenius root of A.*
2. *Any positive eigenvectors of A is the constant multiples of w.*
3. *The absolute values of the eigenvalues of A, except for λ_A, are smaller than λ_A.*
4. *The Frobenius root of the transposed matrix A' is equivalent to the Frobenius root of A.*

This theorem ensures the existence of weight vector in a pairwise comparison matrix.

From Theorem 1, the following theorem about a perturbed comparison matrix holds[7].

Theorem2 *Let $A = (a_{ij})$, $i,j = 1,\ldots,n$ be a comparison matrix and let $A(\varepsilon) = A + \varepsilon D_A$, $D_A = (a_{ij}d_{ij})$ be a matrix that has been perturbed.*
Moreover, let λ_A be the Frobenius root of A, w_1 be the eigenvector corresponding to it, and w_2 be the eigenvector corresponding to the Frobenius root of A', then, the Frobenius root $\lambda(\varepsilon)$ of $A(\varepsilon)$ and the corresponding eigenvector $w_1(\varepsilon)$ can be expressed as follows

$$\lambda(\varepsilon) = \lambda_A + \varepsilon\lambda^{(1)} + o(\varepsilon), \quad (4)$$

$$w_1(\varepsilon) = w_1 + \varepsilon w^{(1)} + o(\varepsilon), \quad (5)$$

where

$$\lambda^{(1)} = \frac{w_2' D_A w_1}{w_2' w_1}, \quad (6)$$

$w^{(1)}$ is an n-dimension vector that satisfies

$$(A - \lambda_A I)w^{(1)} = -(D_A - \lambda^{(1)} I)w_1, \quad (7)$$

where $o(\varepsilon)$ denotes an n-dimension vector in which all components are $o(\varepsilon)$.

(Proof)
From 1 in Theorem 1, the Frobenius root λ_A is the simple root. Thus, expansions (4) and (5) are valid. Therefore, characteristic equations become

$$(A + \varepsilon D_A)(w_1 + \varepsilon w^{(1)} + o(\varepsilon)) \\ = (\lambda_A + \varepsilon\lambda^{(1)} + o(\varepsilon))(w_1 + \varepsilon w^{(1)} + o(\varepsilon)),$$

$$Aw_1 = \lambda_A w_1.$$

From these two equations, (7) can be obtained. Further, from 4 in Theorem 1, $w_2'A = \lambda_A w_2'$ holds, and it becomes

$$w_2' w_1 \lambda^{(1)} = w_2' D_A w_1.$$

Thus, equation (6) holds. (Q.E.D)

3.1 Sensitivity analysis of the consistency index

About a fluctuation of the consistency index, the following corollary can be obtained from Theorem2.

Corollary1 *Using an appropriate g_{ij}, we can represent the consistency index C.I.(ε) of the perturbed comparison matrix as follows*

$$\text{C.I.}(\varepsilon) = \text{C.I.} + \varepsilon \sum_i^n \sum_j^n g_{ij} d_{ij} + o(\varepsilon). \quad (8)$$

(Proof)
From the definition of the consistency index (3) and (4),

$$\text{C.I.}(\varepsilon) = \text{C.I.} + \varepsilon \frac{\lambda^{(1)}}{n-1} + o(\varepsilon)$$

holds. Here, let $w_1 = (w_{1i})$ and $w_2 = (w_{2i})$, from (6), $\lambda^{(1)}$ is represented as

$$\lambda^{(1)} = \frac{1}{w_2' w_1} \sum_i^n \sum_j^n w_{2i} a_{ij} w_{1j} d_{ij},$$

so the second part of the right side is expressed by a linear combination of d_{ij}. (Q.E.D)

To see g_{ij} in the equation (8) in Corollary 1, how the components of a comparison matrix impart influence on its consistency can be found.

On the other hand, since the comparison matrix

$A(\varepsilon) = (a_{ij}(\varepsilon))$ is reciprocal; $a_{ji}(\varepsilon) = 1/a_{ij}(\varepsilon)$ holds, and it becomes

$$a_{ji} + \varepsilon a_{ji} d_{ji} = \frac{1}{a_{ij}} - \varepsilon \frac{d_{ij}}{a_{ij}} + o(\varepsilon). \quad (9)$$

Here, since $a_{ij} = 1/a_{ji}$,

$$d_{ji} = -d_{ij} \quad (10)$$

is obtained. In fact, we can see influence more easily by use of this property.

3.2 Sensitivity analysis of weights

About the fluctuation of the weight, the following corollary also can be obtained from Theorem 2.

Corollary 2 *Using an appropriate $h_{ij}^{(k)}$, we can represent the fluctuation $w^{(1)}=(w_k^{(1)})$ of the weight (i.e. the eigenvector corresponding to the Frobenius root) as follows*

$$w_k^{(1)} = \sum_i^n \sum_j^n h_{ij}^{(k)} d_{ij}. \quad (11)$$

(Proof)
The k-th row component of the right side of (7) in Theorem 2 is represented as

$$\sum_i^n \sum_j^n \{ \frac{w_{1k} w_{2i} a_{ij} w_{1j}}{w_2 w_1} - \delta(i,k) a_{ij} w_{1j} \} d_{ij},$$

and is expressed by a linear combination of d_{ij}. Here, $\delta(i,k)$ is Kronecker's symbol

$$\delta(i,k) = \begin{cases} 1 & (i = k), \\ 0 & (i \neq k). \end{cases}$$

On the other hand, since λ_A is a simple root, Rank$(A - \lambda_A I) = n$-1. Accordingly, the weight vector is normalized as

$$\sum_k^n (w_{1k} + \varepsilon w_k^{(1)}) = \sum_k^n w_{1k} = 1,$$

then the following condition follows.

$$\sum_k^n w_k^{(1)} = 0. \quad (12)$$

Using elementary transformation to formula (7) in the condition above, we also can represent $w_k^{(1)}$ by linear combinations of d_{ij}. (Q.E.D)

From the equation (5) in Theorem 2, the component that has a great influence on weight $w_1(\varepsilon)$ is the component which has the greatest influence on $w^{(1)}$. Accordingly, from Corollary 2, how components of a comparison matrix impart influence on the weights, can be found, to see $h_{ij}^{(k)}$ in the equation (11).

Of course, we can also see influence more easily by use of equation (10).

4 Representation of weights using fuzzy sets

It often occurs that a comparison matrix has not so good consistency (i.e. 0.1<C.I.<0.2), since there are many activities. In these cases, we consider that components of a comparison matrix have fuzziness, because they are results from fuzzy judgment of human. Therefore weights could be treated as fuzzy numbers.

Here, to represent fuzziness of weight w_{1i}, we use L-R fuzzy number.

4.1 L-R fuzzy number

L-R fuzzy number

$$M = (m, \alpha, \beta)_{LR}$$

is defined as fuzzy sets whose membership function is as follows.

$$\mu_M(x) = \begin{cases} R\left(\frac{x-m}{\beta}\right) & (x > m), \\ L\left(\frac{m-x}{\alpha}\right) & (x \leq m). \end{cases}$$

Where $L(x)$ and $R(x)$ are shape function which satisfies

(1) $L(x) = L(-x)$,
(2) $L(0) = 1$,
(3) $L(x)$ is non increasing function.

4.2 Fuzzy weight

The multiple of coefficients $g_{ij} h_{ij}^{(k)}$ in Corollary 1 and 2 is considered as the influence concerned with a_{ij}, from the fluctuation of the consistency index.

Since g_{ij} is always positive, if the coefficient $h_{ij}^{(k)}$ is positive, the real weight of activity k is considered as bigger than w_{1k}, and if $h_{ij}^{(k)}$ is negative, the real weight of activity k is considered as smaller. Therefore, a sign of $h_{ij}^{(k)}$ represents a direction the spread of the fuzzy number. Of course the absolute value $g_{ij} |h_{ij}^{(k)}|$ represents the amount of the influence.

On the other hand, C.I. becomes bigger then the judgment becomes more fuzziness.

Consequently, multiple C.I. $g_{ij}|h_{ij}^{(k)}|$ can be regarded as a spread of a fuzzy weight $\tilde{v}_k$ concerned with a_{ij}.

Definition 1 (fuzzy weight) *Let w_{1k} be a crisp weight of activity k, and g_{ij}, $|h_{ij}^{(k)}|$ denote the coefficients found in Corollary 1 and 2. When 0.1<C.I.<0.2 holds, a fuzzy weight $\tilde{v}_k$ is defined by*

$$\tilde{v}_k = (w_{1k}, p_k, q_k)_{LR}, \quad (13)$$

where

$$p_k = \mathrm{C.I.}\sum_i^n \sum_j^n s(-,h_{ij}^{(k)}) g_{ij} |h_{ij}^{(k)}|, \quad (14)$$

$$q_k = \mathrm{C.I.}\sum_i^n \sum_j^n s(+,h_{ij}^{(k)}) g_{ij} |h_{ij}^{(k)}|, \quad (15)$$

$$s(+,h) = \begin{cases} 1 & (h > 0), \\ 0 & (h \leq 0), \end{cases}$$

$$s(-,h) = \begin{cases} 0 & (h > 0), \\ 1 & (h \leq 0). \end{cases}$$

We can calculate the fuzzy weights of activities using Definition 1. Then, the fuzzy weights of alternatives are calculated by operations of fuzzy numbers. They show how the result from AHP has fuzziness.

5 Experiment

In this section, we introduce an experiment about an employment selection problem, whose structure is shown in Figure 1.

Table 2 shows a comparison matrix of activities, and Table 3 shows weights of alternatives with respect to activities.

Calculated weights of activities are as follows.

$$\begin{pmatrix} \text{Salary} \\ \text{Stability} \\ \text{Prospect} \\ \text{Attractive} \\ \text{Holiday} \\ \text{Welfare} \end{pmatrix} = \begin{pmatrix} 0.04 \\ 0.41 \\ 0.26 \\ 0.16 \\ 0.04 \\ 0.09 \end{pmatrix}$$

However, they do not have much reliability, because consistency of the matrix is not so good (C.I.=0.13).

Table 4 shows a result of sensitivity analysis of consistency. The comparison value between salary and stability, or the value between holiday and welfare has much influence.

Table 5 shows fuzzy weights of activities. A weight of stability has most fuzziness.

Table 2: A comparison matrix of activities

	Salary	Stability	Prospect	Attractive	Holiday	Welfare
Salary	1	1/5	1/5	1/5	1/2	1/3
Stability		1	3	4	7	5
Prospect			1	3	6	5
Attractive				1	7	3
Holiday					1	1/5
Welfare						1

Table 3: Weights of alternatives with respect to activities.

	Company A	Company B	Company C
Salary	0.158	0.766	0.076
Stability	0.042	0.180	0.778
Prospect	0.180	0.778	0.042
Attractive	0.070	0.751	0.178
Holiday	0.157	0.249	0.594
Welfare	0.121	0.115	0.764

Table 4: Result of sensitivity analysis of consistency

	Salary	Stability	Prospect	Attractive	Holiday	Welfare
Salary	0.0000	0.3602	0.2291	0.1414	0.0847	0.1279
Stability	0.0789	0.0000	0.3254	0.2677	0.1123	0.1817
Prospect	0.1199	0.0864	0.0000	0.3052	0.1483	0.2781
Attractive	0.1907	0.1031	0.0874	0.0000	0.2715	0.2635
Holiday	0.3394	0.2620	0.1945	0.1028	0.0000	0.0782
Welfare	0.2131	0.1536	0.0977	0.1006	0.3613	0.0000

Table 5: fuzzy weights of activities

	Center	Spread(L)	Spread(R)
Salary	0.04	0.0011	0.0012
Stability	0.41	0.0065	0.0069
Prospect	0.26	0.0059	0.0059
Attractive	0.16	0.0026	0.0040
Holiday	0.04	0.0013	0.0013
Welfare	0.09	0.0022	0.0023

On the other hand, priorities of alternatives (i.e. composite weights) of Company B is not different from that of company C as follows

$$\begin{pmatrix} \text{Company A} \\ \text{Company B} \\ \text{Company C} \end{pmatrix} = \begin{pmatrix} 0.10 \\ 0.45 \\ 0.45 \end{pmatrix}.$$

Here, by use of fuzzy weights of activities, composite weights are represented as in Figure 2. Then, we can see how fuzziness they have, and can get a key of interpretation. In this example, priority of Company B seems to be little higher than Company C.

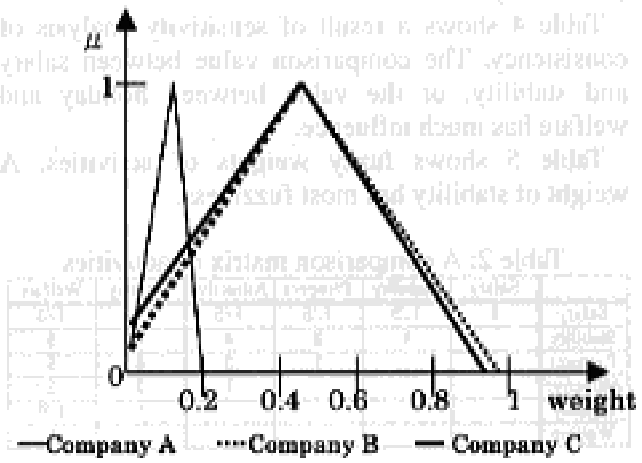

Figure 2: composite weights using fuzzy weight

6. Conclusions

We proposed one representation of the weights of AHP by use of fuzzy sets and the result of the sensitivity analysis for the cases that consistency of a comparison matrix is not so good. Moreover, through a numerical example, we showed not only how to represent weights, but also how the result of the AHP has fuzziness, when a little bad consistency exists.

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